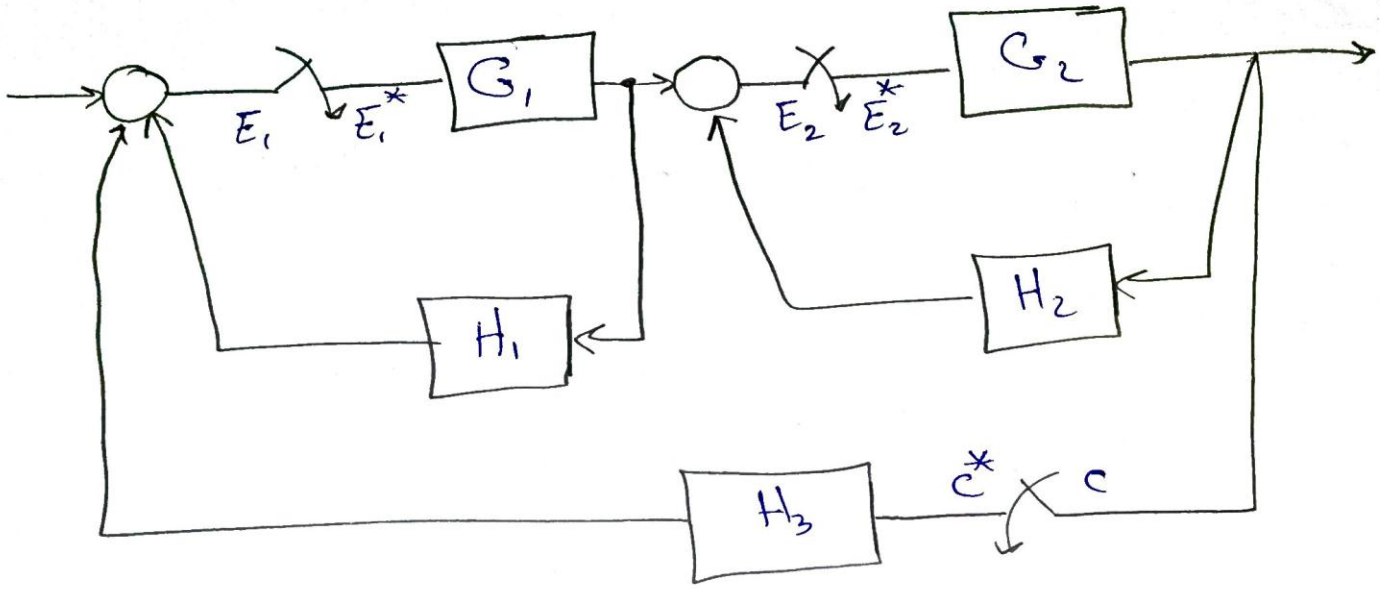
Digital control Lec 3



steady state error (s.s.e)

$e(t) \longrightarrow$ s.s.e : the error at $t \longrightarrow \infty$

$$s.s.e = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s)$$

$$= \lim_{z\to 1} (1-z^{-1})\,E(z) = \lim_{z\to 1} (z-1)\,E(z)$$

Cont. time

$$r(t) = 1 \text{ (unit step)} \longleftrightarrow s.s.e = \frac{1}{1+K_p}$$

$$; K_p = \lim_{s\to 0} GH(s)$$ "Position error Constant"

② $r(t) = t$ "unit ramp"

$$\rightarrow s.s.e = \frac{1}{K_v} \Rightarrow K_v = \lim_{s \to 0} s\,G H(s)$$

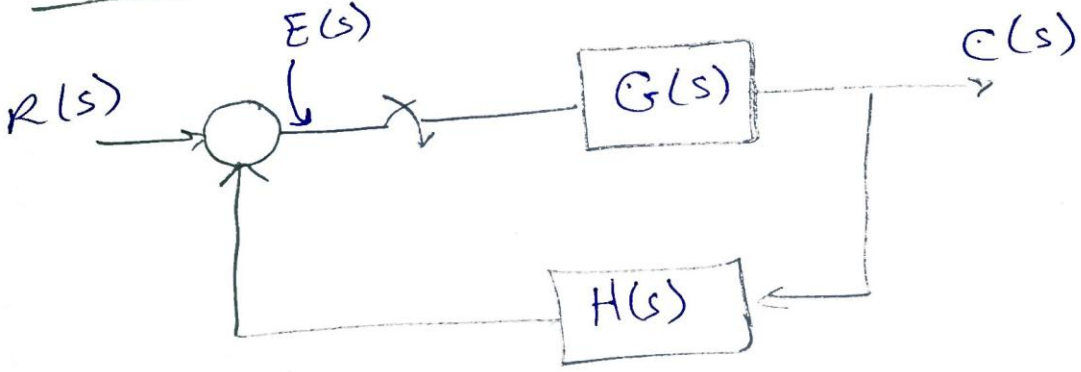
$K_v \rightarrow$ velocity error constant

③ $r(t) = \frac{t^2}{2}$

$$\rightarrow s.s.e = \frac{1}{K_a} \Rightarrow K_a = \lim_{s \to 0} s^2\,G H(s)$$

$K_a \equiv$ acceleration error const.

*steady state error in discrete-time system



→ The general form for discrete-time feedback

$$s.s.e = \lim_{s \to 0} s\,E(s) = \lim_{z \to 1} (z-1)\,E(z)$$

in terms of $R(z)$

$$C(s) = G E^*$$

$$E(s) = R(s) - GHE^* \longrightarrow E^* = R^* - \overline{GH}^* E^*$$

$$(1 + \overline{GH}^*) E^* = R^*$$

$$E^* = \frac{R^*}{1 + \overline{GH}^*} \Rightarrow E(z) = \frac{R(z)}{1 + \overline{GH}(z)}$$

$$s.s.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

s.s.e depends on:

1) system i/p $\Rightarrow R(z)$

2) Digital o.L.T.F $= \overline{GH}(z)$ or system type.

The general form for Digital o.L.T.F

$$\overline{GH}(z) = \frac{N(z)}{(z-1)^n D(z)}$$

$n \longrightarrow$ system type

or

$$\overline{GH}(z) = \frac{N(z)}{(1-z^{-1})^n D(z)}$$

والله نفس المعنى

① For unit-step i/p

$$r(t) = 1 \Rightarrow r(KT) = 1$$

$$\boxed{R(z) = \frac{z}{z-1}}$$

$$S.S.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1 + \overline{GH(z)}}$$

$$= \lim_{z \to 1} (z-1) \frac{z}{(z-1)(1 + \overline{GH(z)})}$$

$$= \lim_{z \to 1} \frac{z}{1 + \overline{GH(z)}}$$

نوزع الـ (lim) عالبسط والمقام.

$$S.S.e = \frac{1}{1 + \lim_{z \to 1} \overline{GH(z)}} = \frac{1}{1 + K_p}$$

$$K_p = \lim_{z \to 1} \overline{GH(z)}$$

where $K_p \rightarrow$ Position error Constant.

[2] For unit ramp

$$r(t) = t \Rightarrow r(KT) = KT$$

$$R(z) = \frac{Tz}{(z-1)^2}$$

$$S.S.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1 + \overline{GH(z)}}$$

$$= \lim_{z \to 1} (z-1) \frac{Tz}{(z-1)^2 (1 + \overline{GH(z)})}$$

$$= \lim_{z \to 1} \frac{Tz}{(z-1) + (z-1)\overline{GH(z)}} = \frac{T}{0 + \lim_{z \to 1} (z-1)\overline{GH(z)}}$$

لم نقوف في آخر حد لأنه من الممكن أن تحتوي $\overline{GH(z)}$ على $(z-1)$ في المقام فتلغى $(z-1)$.

$$S.S.e = \frac{1}{\frac{1}{T} \lim_{z \to 1} (z-1) GH(z)} = \frac{1}{K_v}$$

$K_v \rightarrow$ velocity error constant.

$$K_v = \frac{1}{T} \lim_{z \to 1} (z-1) GH(z) \Rightarrow$$

[3] For acceleration i/p

$$r(t) = \frac{t^2}{2}$$

Note $\quad t \xrightarrow{Z.T} \frac{TZ}{(Z-1)^2}$

$$Z\left(t \cdot f(t)\Big|_{t=KT}\right) = -TZ\,\frac{dF(z)}{dz}$$

$$Z[t \cdot t] = -TZ\,\frac{d}{dz}\left(\frac{TZ}{(Z-1)^2}\right)$$

$$= -T^2 Z\left[\frac{(Z-1)^2(1) - Z(2)(Z-1)}{(Z-1)^4}\right] = -T^2 Z\left[\frac{(Z-1) - 2Z}{(Z-1)^3}\right]$$

$$= -T^2 Z\,\frac{-Z-1}{(Z-1)^3} = \frac{T^2 Z(Z+1)}{(Z-1)^3}$$

$$t^2 \xrightarrow{Z.T} \frac{T^2 Z(Z+1)}{(Z-1)^3}$$

$$\frac{t^2}{2} \xrightarrow{Z.T} \frac{\frac{T^2}{2} Z(Z+1)}{(Z-1)^3}$$

$$S.S.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1 + \overline{GH(z)}}$$

$$= \lim_{z \to 1} (z-1) \frac{\frac{T^2}{2} z(z+1)}{(z-1)^3 \left(1 + \overline{GH(z)}\right)}$$

$$= \lim \frac{\frac{T^2}{2} z(z+1)}{(z-1)^2 + (z-1)^2 \overline{GH(z)}}$$



$$= \frac{\lim_{z \to 1} \frac{T^2}{2} z(z+1)}{\frac{d}{T^2} \lim_{z \to 1} (z-1)^2 + \lim_{z \to 1} (z-1)^2 \overline{GH(z)}}$$

$$= \frac{1}{\frac{1}{T^2} \lim (z-1)^2 \overline{GH(z)}}$$

$$K_a = \frac{1}{T^2} \lim (z-1)^2 \overline{GH(z)}$$

Where: $K_a \rightarrow$ acceleration error Constant

$$S.S.e = \frac{1}{K_a}$$

$$r(t)=1 \Big|_{t=KT} \longrightarrow s.s.e = \frac{1}{1+K_p} \Rightarrow K_p = \lim_{z\to 1} \overline{GH(z)}$$

$$r(t)=t \Big|_{t=KT} \longrightarrow s.s.e = \frac{1}{K_v} \Rightarrow K_v = \lim_{z\to 1} (z-1)\,\overline{GH(z)}$$

$$r(t)=\frac{t^2}{2} \Big|_{t=KT} \longrightarrow s.s.e = \frac{1}{K_a} \Rightarrow K_a = \frac{1}{T^2} \lim_{z\to 1} (z-1)^2\,\overline{GH(z)}$$

type \ I/P	unit-step $r(t)=1$	unit ramp $r(t)=t$	acceleration i/p $r(t)=\frac{t^2}{2}$
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$

s.s.e depends on:

i) i/p

ii) type of i/p.

Ex1

① $\overline{GH(z)} = \dfrac{-10}{(z-1)(1-z)(z^2-1.4z+1)} = \dfrac{10}{(z-1)^2(z^2-1.4z-1)}$

② $\overline{GH(z)} = \dfrac{z+1}{(z-1)(z-2)}$

③ $\overline{GH(z)} = \dfrac{z+0.5}{(1-z)^2(z^2-2)}$

Find a) system type and order.

b) error constants.

c) s.s.e for $r(t)=1$ & $r(t)=t$ ~~& $r(t)=$~~

$\& r(t) = 2+3t+4t^2$

order ← أعلى أس في المقام

[1] a) type = 2 & order = 4

b)

$K_p = \lim_{z\to 1} \overline{GH(z)} = \infty$

$K_v = \frac{1}{T} \lim_{z\to 1} (z-1)\, GH(z)$ (assume $T=1$ sec)

$K_v = \infty$

$$K_a = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 \overline{GH}(z)$$

$$= \frac{10}{(z^2 - 1.4z + 1)} = \frac{10}{0.6}$$

c)

$$r(t) = 1 \longrightarrow s.s.e = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

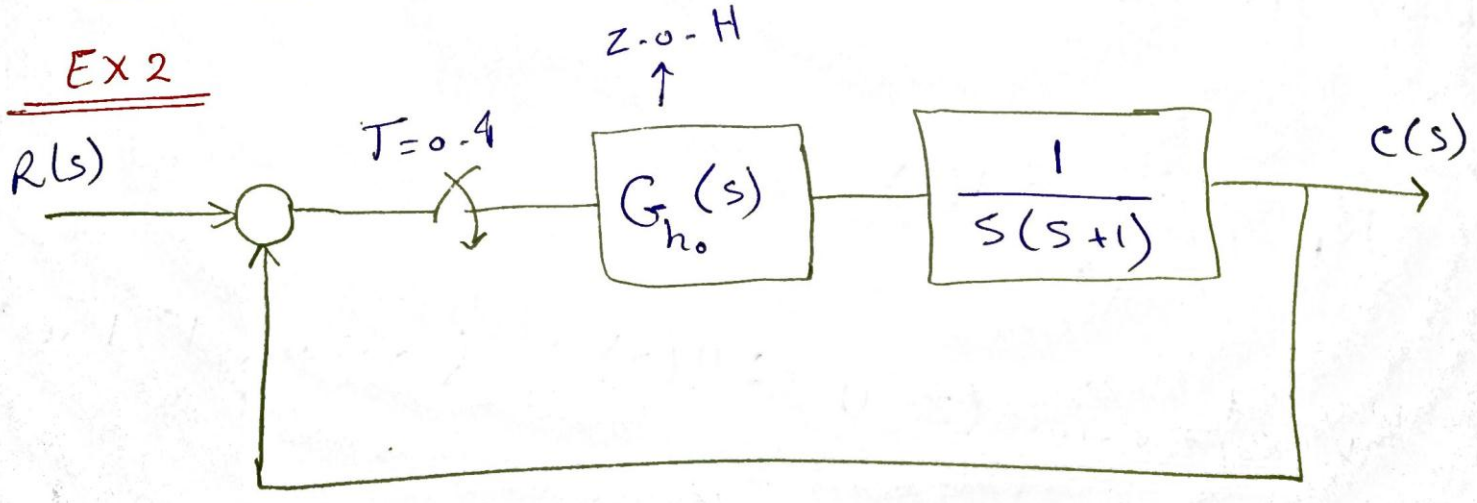
$$r(t) = t \longrightarrow s.s.e = \frac{1}{K_v} = \frac{1}{\infty} = 0$$

$$r(t) = 2 + 3t + 4t^2 \rightarrow \frac{8t^2}{2}$$

$$s.s.e = 2\left(\frac{1}{1+K_p}\right) + 3\left(\frac{1}{K_v}\right) + 8\left(\frac{1}{K_a}\right)$$

$$= 8\left(\frac{0.6}{10}\right) = 0.48$$

EX 2



Find

① o.L. Pulse T.F.

② C.L Pulse T.F

③ type and order of the system

④ S.S.e for unit step i/p and unit ramp i/p.

⑤ steady state value of the o/p for unit step i/p.

⑥ Find the first 3 terms of o/p sequence

$y(0), y(0.4), y(0.8)$ for unit step i/p.

$y(KT)\big|_{K=0}$ $y(KT)\big|_{K=1}$ $y(KT)\big|_{K=2}$

لو عندنا في الرسم $H(s)$

$$o.L.T.F = \overline{G\,H(z)} \Rightarrow G(z)$$

$$C.L.T.F = \frac{G(z)}{1+\overline{G\,H(z)}} \Rightarrow \frac{G(z)}{1+G(z)}$$

Sol.

For o.L.T.F

$$o.L.T.F = G(z) = Z\left[\frac{1-e^{-Ts}}{s^2(s+1)}\right]$$

$$O.L.T.F = (1-z^{-1})\ Z\left[\frac{1}{s^2(s+1)}\right]$$

$$= (1-z^{-1})\ Z\left[\frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{s+1}\right]$$

$$A_1 = 1\ ,\quad A_3 = 1\ ,\quad A_2 = -1$$

$$O.L.T.F = (1-z^{-1})\ Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right]$$

$\frac{1}{s^2} \rightarrow t$, $\frac{1}{s} \rightarrow 1$, $\frac{1}{s+1} \rightarrow e^{-t}$

$$= (1-z^{-1})\ Z\left[t\Big|_{t=KT} - 1\Big|_{t=KT} + e^{-t}\Big|_{t=KT}\right]$$

$$= (1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right]$$

$$= \frac{z-1}{z}\left[\frac{0.4\,z}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-0.67}\right]$$

$$= \frac{0.4}{z-1} - 1 + \frac{z-1}{z-0.67}$$

$$= \frac{0.4(z-0.67)-(z-1)(z-0.67)+(z-1)^2}{(z-1)(z-0.67)}$$

$$G(z) = \frac{0.07z + 0.062}{(z-1)(z-0.67)}$$

open loop Pulse T.F

or open loop digital T.F

② C.L.T.F $= \frac{G(z)}{1+GH(z)} = \frac{G(z)}{1+G(z)}$

$$= \frac{\frac{0.07z+0.062}{(z-1)(z-0.67)}}{1+\frac{0.07z+0.062}{(z-1)(z-0.67)}}$$

$$= \frac{0.07z+0.062}{(z-1)(z-0.67)+0.07z+0.062}$$

$$C.L.T.F = \frac{0.07z+0.062}{z^2-1.6z+0.732}$$

③

type = 1 & order = 2 (2nd order system)

④ s.s.e for unit step & unit ramp

$$K_p = \lim_{z \to 1} \overline{GH(z)} = \lim_{z \to 1} \overline{G(z)} = \infty$$

$$\text{s.s.e}\Big|_{r(t)=1} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

$$K_v = \frac{1}{T} \lim_{z \to 1} (z-1)\,\overline{GH(z)} = \frac{1}{T} \lim_{z \to 1} (z-1)\,G(z)$$

$$= \frac{1}{0.4}\left(\frac{0.07 + 0.062}{1 - 0.67}\right) = 1$$

$$\text{s.s.e}\Big|_{r(t)=t} = \frac{1}{K_v} = 1$$

⑤ $\underline{c(\infty)} = \lim_{z \to 1} (z-1)\,C(z)$

Final value

↳ (steady state value for o/p)

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}$$

$$C(z) = (T.F)\ R(z)$$

$$C(z) = \left[\frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}\right] R(z)$$

For unit step i/p $\left(R(z) = \frac{z}{z-1}\right)$

$$C(z) = \frac{(0.07z + 0.062)z}{(z-1)(z^2 - 1.6z + 0.732)}$$

$$c(\infty) = \lim_{z \to 1} (z-1)\ C(z)$$

$$= \lim \frac{(0.07z + 0.062)z}{z^2 - 1.6z + 0.732} = \boxed{1}$$

→ another solution for point (5)

$$e(t) = r(t) - y(t)$$

$$e(\infty) = r(\infty) - y(\infty)$$

$e(\infty)$ → S.S.e ; $y(\infty)$ → steady state o/p.

$$y(\infty) = r(\infty) - e(\infty)$$ f.r unit step i/p

$$y(\infty) = 1$$ Cause $r(\infty) = 1$, $e(\infty) = 0$

. C في المعادلة في y ← ي

⑥ Find $c(0)$ & $c(0.4)$ & $c(0.8)$

$$C(z) = (T.F)\, R(z)$$

$$= \left[\frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}\right]\left(\frac{z}{z-1}\right)$$

$$= \frac{0.07z^2 + 0.062z}{z^3 - 2.6z^2 + 2.33z - 0.732}$$

$$z^3 - 2.6z^2 + 2.33z - 0.732 \;\overline{)\; 0.07z^2 + 0.062z \;}\; 0.07z^{-1} + 0.244z^{-2}$$

$$0.07z^2 - 0.182z + 0.1631 - 0.0510z^{-1}$$

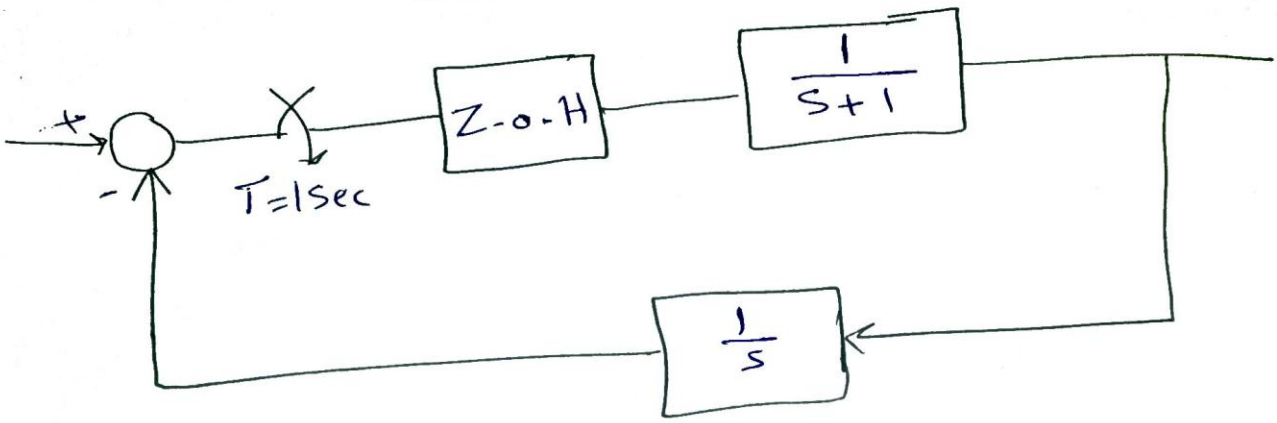
$$0.244z - 0.163 + 0.0510z^{-1}$$

$$C(z) = 0.07z^{-1} + 0.244z^{-2} + \cdots$$

$$= c(0) + c(T)z^{-1} + c(2T)z^{-2} \cdots$$

$$c(0) = 0,\ c(0.4) = 0.07\ ;\ c(0.8) = 0.244$$

Report



→ Find the unit step response

⇝ and discuss the system stability.

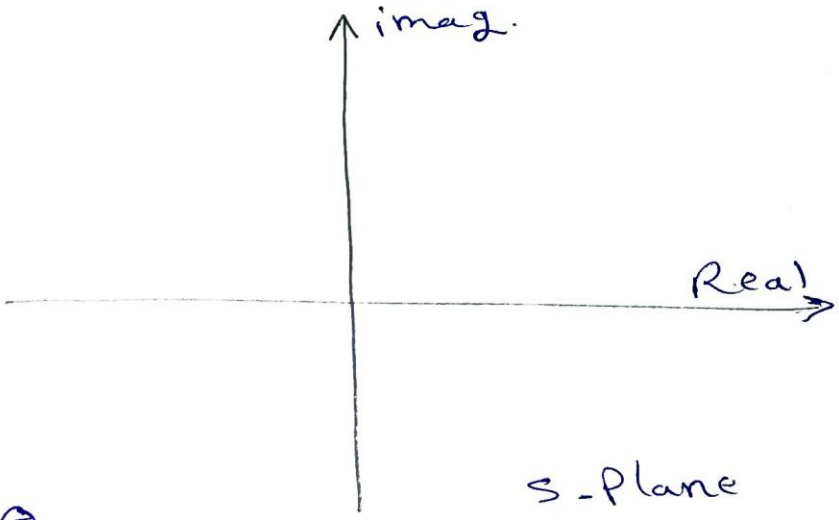
ـــ ال (system) و الشكل السابق (stable)

لأن ال i/p = 1 والخرج = 1 (Final value)

Relation Z-domain and s-domain

$$Z = e^{+TS}$$

for: $s = \underbrace{\sigma}_{\text{real}} + \underbrace{j\omega}_{\text{imag. part}}$



$$Z = e^{T(\sigma + j\omega)}$$

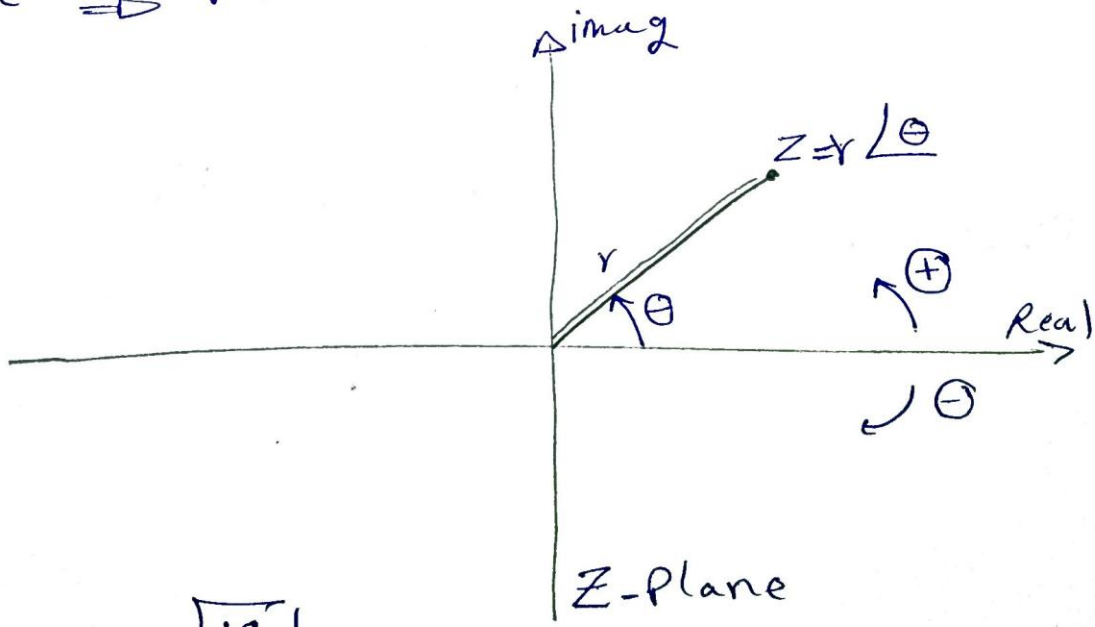
$$= e^{\sigma T + j\omega T}$$

$$Z = e^{\sigma T} \cdot e^{j\omega T}$$

$$= 1 \angle\, = r \angle \theta$$

$$Z = e^{\sigma T} \angle \omega T$$

for $T = 1$ sec $\Rightarrow r = e^{\sigma}$ & $\theta = \omega$



$|z| < 1$

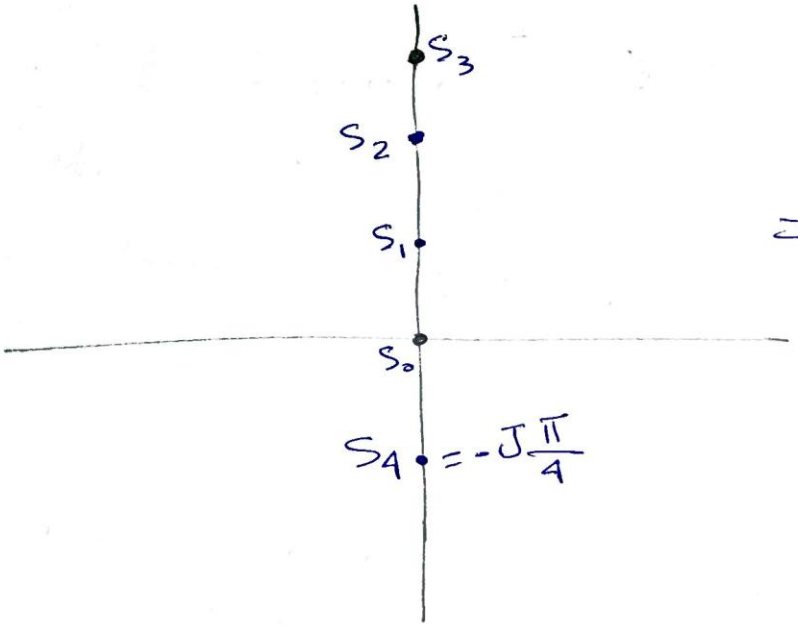
$S_0 = 0 + J0$

$S_0 \longrightarrow r \angle \theta$

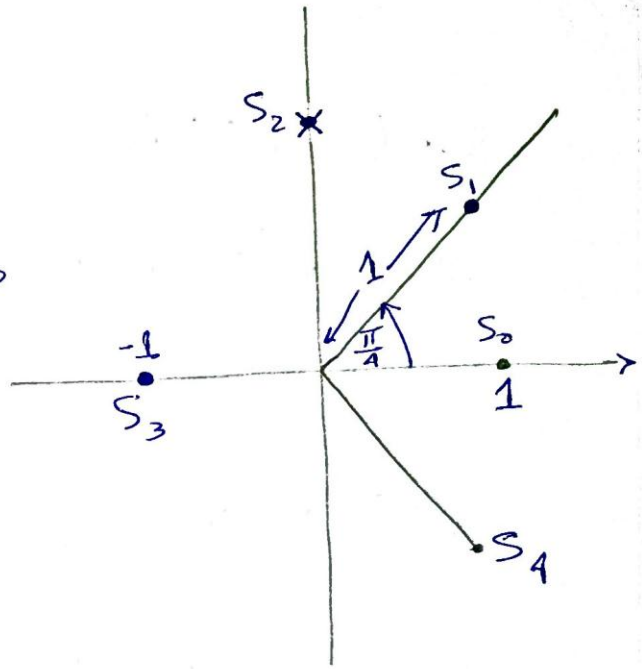
$r = e^{0}$

$\theta = 0$

$= 1 \angle 0$



S-plane



Z-plane

$S_1 = 0 + J\frac{\pi}{4}$

$S_1 \longrightarrow r = e^{0} \angle \frac{\pi}{4} = 1 \angle 45$

$S_2 = 0 + J\frac{\pi}{2}$

$S_2 \longrightarrow 1 \angle \frac{\pi}{2}$

$S_3 = J\pi \longrightarrow 1 \angle \pi$

← لو حولنا (mapping) لـ (imag. axis) في الـ (s-plane)

الى (z-plane) ينتج دائرة نصف قطرها 1 ومركزها

نقطة الأصل.

← في الـ (s-plane) في حساب الـ (stability) بيكون

على حسب الـ Imag. أما في الـ (z-plane) بيكون التقدير

طبقاً للدائرة التي نصف قطرها 1.

stability check

1) $\tilde{e}^{\sigma}$ ($\sigma < 0$ (-ve)) $\equiv \tilde{e}^{\sigma} < 1$ ⇒ stable

$|z| = r = \tilde{e}^{\sigma} < 1$ (stable)

(Pole lies in unit circle → stable Pole)

2) $\tilde{e}^{\sigma}$ ($\sigma > 0$ (+ve)) $\equiv \tilde{e}^{\sigma} > 1$ (un stable)

$|z| = r = \tilde{e}^{\sigma} > 1$

(Pole lies outside unit circle → unstable pole)

3)

$\sigma = 0 \longrightarrow e^{0} = 1$ ⇝ Critical Pole

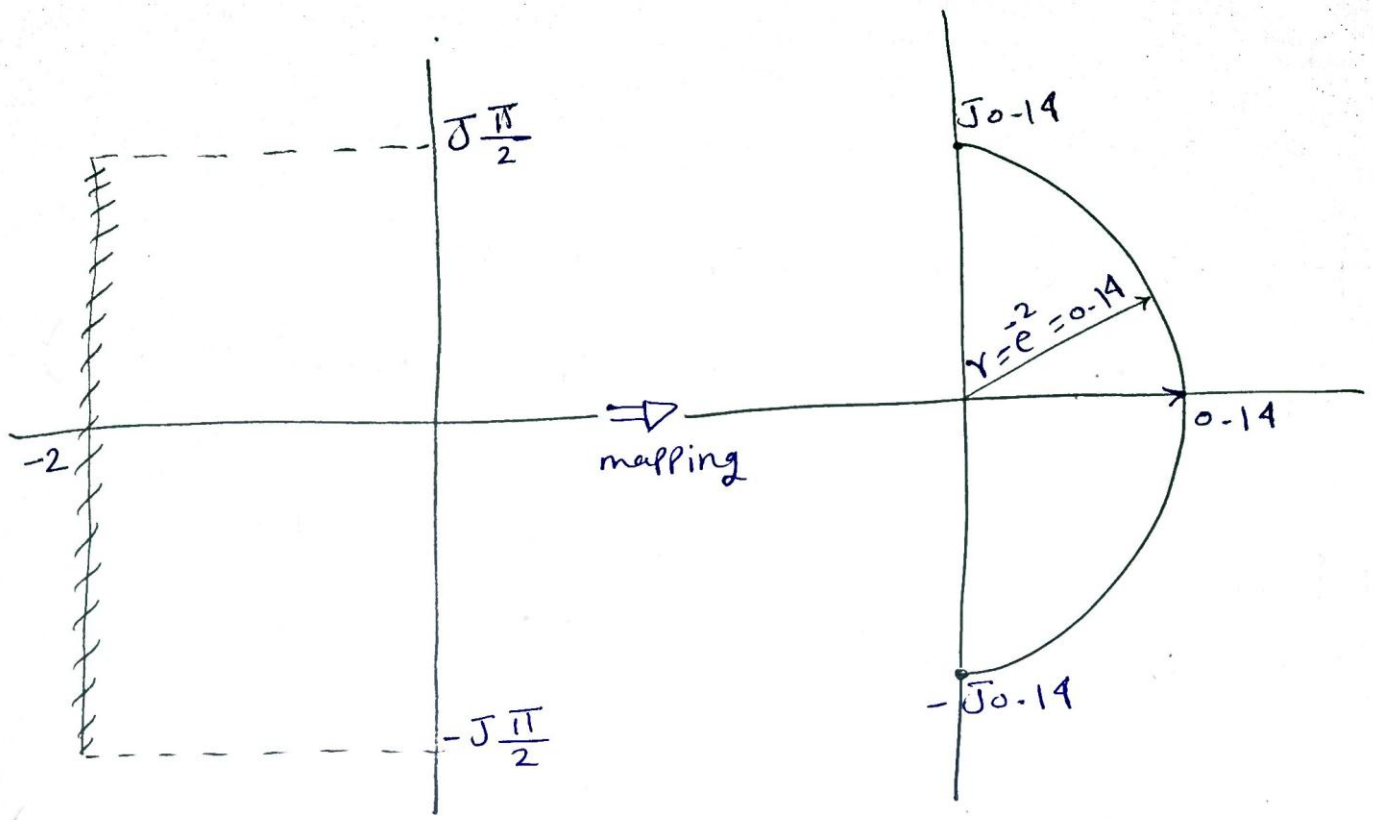
$|z| = r = e^{\sigma} = 1$ (Critical Pole)

⇝ Pole lines on unit circle ⇒ Critical Pole)

1. The system is stable, if all Poles lies inside unit circle ($|z_p| < 1$)

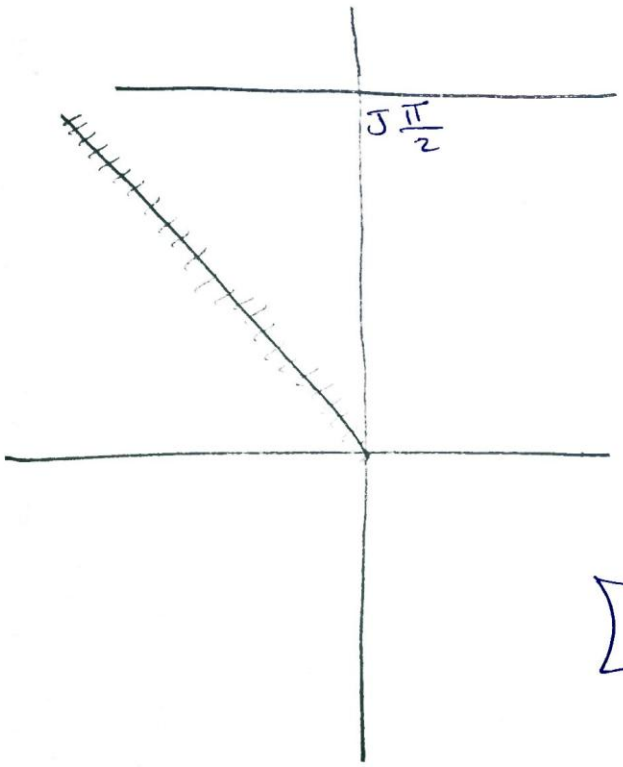
2. The system is unstable, if at least one Pole lies outside unit circle ($|z| > 1$)

3. The system is critical stable, if one ore more of Poles lies on the unit circle and the other Poles lies inside the unit circle.



(s-Plane)

(z-Plane)



← Report